647510 **MSC4MatC401x** **Seat No :________**

M.Sc.(Maths.) Semester - 4 (*CBCS*) Examination
March/April - 2025 (NEW COURSE)
Linear Algebra (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Que1(A) Answer any two questions out of three. (10)

(1) Let $\lambda \in F$ be a characteristic root of $T \in A(V)$, then prove that λ is a root of the minimal polynomial of T. In particular, T only has a finite number of characteristic roots in F.

(2) Let V be a finite dimensional vector space over F. Show that $T \in A(V)$ is regular if and only if T maps V onto V.

(3) Let $\mathcal{A}$ be an algebra over F with unit element. Prove that $\mathcal{A}$ is isomorphic to a subalgebra of $A(V)$, for some vector space V over F.

Que-1(B) Answer any one question out of two. (04)

(1) Let V b a finite dimensional vector space over F. Prove that $T \in A(V)$ is invertible if and only if the constant term in the minimal polynomial for T is not 0.

(2) Let V be a finite dimensional vector space over F, then for $S, T \in A(V)$ prove that

a) $r(ST) \leq r(T)$

b) $r(TS) \leq r(T)$

c) $r(ST) = r(TS) = r(T)$, for S is regular in $A(V)$

Oue-2(A) Answer any two questions out of three. (10)

(1) Let V be a finite dimensional vector space over F. If $T \in A(V)$ has all its characteristic roots of T are in F then prove that T satisfies a polynomial of degree n over F.

(2) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent with index of nilpotency n_1. Then prove that there is a basis of V in which

the matrix of T is of the form $\begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_n} \end{pmatrix}$

where $n_1 \geq n_2 \geq \cdots \geq n_k$ and $dimV = n_1 + n_2 + \cdots + n_k$.

(3) Let V be a finite dimensional and $T \in A(V)$ be nilpotent, then prove that the set of invariants of T is unique.

Que-2(B) Answer any one question out of two. (04)

(1) Let $u \in V$ be such that $uT^{n_1-k} = 0$, where $0 \leq k \leq n_1$, then show that $u = u_0T^k$, for some $u_0 \in V_1$.

(2) Let $T \in A(V)$ be nilpotent, then show that $\alpha_0 + \alpha_1 T \ldots + \alpha_m T^m$, where $\alpha_i \in F$, is invertible if $\alpha_0 \neq 0$.

Que-3(A) Answer any two questions out of three. (10)

(1) State and prove Jordan decomposition theorem.

(2) Find all possible Jordan forms $T \in A(V)$ having characteristic polynomial $x^2(5 - x)$.

(3) Let $T \in A(V)$ over F. Let $p(x) = \gamma_0 + \gamma_1 x + - \cdots + \gamma_{r-1}x^{r-1} + x^r$ be a minimal polynomial for T over F. Further suppose a module V as a cyclic module. Then prove that there is basis of V over F such that, in that basis,

the matrix of T is $\begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{r-1} \end{pmatrix}$

Que-3(B) Answer any one question out of two. (04)

(1) Let V be a finite dimensional vector space over F and $T \in A(V)$. Let $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let $T_1 = T|v_1$ and $T_2 = T|v_2$. If the minimal polynomial of T_1 and T_2 over F are $p_1(x)$ and $p_2(x)$ then prove that the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

(2) Find all possible Jordan forms $T \in A(V)$ having characteristic polynomial $(x+1)^4$.

Que-4(A) Answer any two questions out of three. (10)

(1) Let $T \in A(V)$, then prove that trT, the trace of T, is the trace of $m_1(T)$, where $m_1(T)$ is the matrix of T in some basis of V.

(2) Let $T \in A(V)$, then prove that $T^* \in A(V)$. Moreover for all $S, T \in A(V)$ and $\alpha \in F$, show that

a) $(T^*)^* = T$

b) $(S+T)^* = S^* + T^*$

c) $(\lambda T)^* = \bar{\lambda} S^*$

d) $(ST)^* = T^* S^*$

(3) For $A, B \in M_n(F)$ prove that $\det(AB) = \det(A)\det(B)$.

Que-4(B) Answer any one question out of two. (04)

(1) Prove that determinants of a matrix and its transpose are same.

(2) Prove that A is invertible if and only if $detA \neq 0$.

Que-5(A) Answer any two questions out of three. (10)

(1) Let V be a finite-dimensional vector space over a field of characteristic zero, and let f be a symmetric bilinear form on V. Then prove that there is an ordered basis for V in which f is represented by a diagonal matrix.

(2) Let V be an n-dimensional vector space over $\mathbb{R}$. Let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $\{\beta_1, \beta_2, \dots, \beta_n\}$ for V in which the matrix of f is diagonal and such that $f(\beta_j, \beta_j) = \pm 1$; $j = 1, \dots, r$. Furthermore, show that the number of basis vectors β_i for which $f(\beta_j, \beta_j) = 1$ is independent of the choice of basis.

(3) Define bilinear form. Let $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for V and $\mathfrak{B}^* = \{L_1, \dots, L_n\}$ be the dual of V^*, then prove that the n^2 bilinear forms $f_{ij}(\alpha, \beta) = L_i(\alpha)L_j(\beta)$; $1 \le i, j \le n$ form a basis for the space $L(V, V, F)$. In particular show that $dim(L(V, V, F)) = n^2$.

Que-5(B) Answer any one question out of two. (04)

(1) Let V be an n-dimensional vector space over $\mathbb{C}$. Let f be a non-degenerate symmetric bilinear form on V. Then the group preserving f is isomorphic to the complex orthogonal group $O(n, \mathbb{C})$.

(2) Let f be a non-degenerate bilinear form on a finite-dimensional vector space V. Prove that the set of all linear operators on V which preserve f is a group under the operation of composition.

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